Limitation on the figure of merit of a quantum frequency discriminator due to fast fluctuations of the rf field

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The influence of the fluctuation characteristics of the rf resonance field on the resolution of a quantum frequency discriminator is investigated. Analytical expressions are obtained for the excess noise introduced by fluctuations of the rf field in the presence of a regular phase modulation, and estimates are presented which show that with present-day technology for generating microwave fields the figure of merit of a quantum discriminator with optical pumping is limited by fluctuations of the rf field at frequencies that are multiples of the modulation frequency. To achieve the limiting values of the figure of merit (limited by the shot noise of the photocurrent) will require microwave sources in which the components in the spectrum of phase–frequency fluctuations at frequencies near twice the modulation frequency are suppressed to a level of −(110–120) dB with respect to the carrier.

INTRODUCTION

The task of achieving the limiting resolution (the "figure of merit" for "quality factor") in a quantum frequency discriminator (QFD) arises in problems of frequency stabilization and quantum magnetometry and is directly related to the total noise level in the signal-detection channel. An important contribution to the noise in the observed signals comes from rapid fluctuations, not eliminated by the automatic fine turning (AFT) system, in the rf field inducing the transitions. This problem is primarily felt in precision microwave discriminators that utilize signals of resonances arising in the populations of the working states; it is most acute in schemes using optical detection, which are characterized by high signal levels.[1-3] For example, the figure of merit of a rubidium frequency discriminator with optical pumping and optical signal detection in the limit imposed by the shot noise of the photocurrent can reach values of 10^3 Hz^{-1} $s^{-1/2}$, but the noise introduced by the rf field formed by sources with a short-term stability of 10^{-12} $s^{-1/2}$ limits the figure of merit to values of the order of 10^2 Hz^{-1} $s^{-1/2}$ (Refs. 3-5). Similarly, the use of optical detection in beam discriminators,[6,7] besides increasing the ratio of the signal to the rms amplitude of the noise in the detection channel, creates a situation in which the figure of merit is limited by fluctuations of the rf field.

The excess noise caused by fluctuations of the rf field in QFDs is a consequence of the use of auxiliary phase (frequency) modulation. The signal corresponding to the difference in the populations of the working levels of the atomic system is given by a quadratic convolution of the acting rf field with the polarization of the working states. Therefore, the total signal contains a contribution from all the combination frequencies of the fluctuation spectrum of the rf field and the regular spectral components of the modulation. The detection system selects the signal component at the modulation frequency; the informative part of the signal is due to the regular deviation (detuning) of the frequency of the rf field from the transition frequency and also to fluctuations of the rf field that lie in a frequency band of ±1/T around the carrier (here T is the time constant of the integrating filter or the dynamic time constant of the AFT ring). This nonlinear transformation causes components of the fluctuation spectrum of the rf field at multiples of the modulation frequency to be carried over into the signal detected at the modulation frequency, forming excess noise. When QFDs are used in frequency-stabilization schemes this noise of the AFT circuit is carried over into low-frequency fluctuations of the rf resonance field, thus lowering the short-term stability of the frequency standard.

In this paper we present a systematic description of the formation of resonance in the populations in a scheme utilizing regular phase modulation of an rf resonance field that contains a fluctuational admixture in its spectrum. We obtain analytical expressions for the excess noise introduced by the rf field and make quantitative estimates of the figures of merit that can be achieved with the existing technology for generating microwave resonance fields.

1. We write the acting rf field in the form

$$U(t)=U_0[1+a(t)]e^{i\omega t+i\varphi(t)}+\text{c. c.},\tag{1}$$

where a(t) and ϕ(t) are real stationary random processes such that

$$E\{a\}=E\{\varphi\}=0,\ E\{a\cdot\varphi\}=0,$$
$$E\{a^2\}=A_a^2,\ E\{\varphi^2\}=A_\varphi^2,\ A_a^2,\ A_\varphi^2\ll 1.\tag{2}$$

We can then write

$$U(t)\simeq U_0[1+z(t)]e^{i\omega t}+\text{c. c.},\tag{3}$$

where $z(t) = a(t) + i\varphi(t)$ is a stationary complex random processes, $E\{z\} = 0$, and $E\{zz^*\} = A_0^2 + A_\varphi^2$.

The quantities z(t), a(t), and ϕ(t) have spectral representations of the form

$$z(t)=\int e^{ixt}ds_z(x),\ a(t)=\int e^{ixt}ds_a(x),\ \varphi(t)=\int e^{ixt}ds_\varphi(x),\tag{4}$$

where x represents random processes with orthogonal increments, and

$$ds_z(x)=ds_a(x)+i\,ds_\varphi(x).\tag{5}$$

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It remains for us to consider the contribution to the zero-frequency converted signal from fast ($|x| > T^{-1}$) fluctuations of the rf field. Neglecting small terms of order $(T\Gamma_1)^{-1}$, $(T\Gamma_2)^{-1}$, we obtain

$$N_{MW}^2 \equiv E\{A_N^2\} = \frac{16B^2}{\Omega^2+\Gamma_1^2}\sum_{p=1}^{\infty} Q_p^2\left\{\left[\sum_m \frac{m\Omega}{\Gamma_2^2+m^2\Omega^2}(J_m J'_{m-p} + J'_m J_{m-p})\right]^2 + \left[\sum_{m'} \frac{\Gamma_2}{\Gamma_2^2+m'^2\Omega^2}(J_{m'} J'_{m'-p} - J'_{m'} J_{m'-p})\right]^2\right\}, \tag{16}$$

where $I'_n = dJ'_n(q)/dq = (J_{n-1} - J_{n+1})/2$;

$$Q_p^2 = \begin{cases} \int_{p\Omega-1/T}^{p\Omega+1/T} dS_a(x) & p = 2N-1,\ N = 1, 2, \ldots, \\ \int_{p\Omega-1/T}^{p\Omega+1/T} dS_\varphi(x) & p = 2N,\ N = 1, 2, \ldots. \end{cases}$$

If $S'_a(x)$ and $S_\Phi'(x)$ for $|x| \gg T^{-1}$ vary slowly on the scale of $2/T$, we can write

$$Q_p^2 = \begin{cases} 2\cdot T^{-1}\cdot S'_a(p\Omega) & p = 2N-1, \\ 2\cdot T^{-1}\cdot S'_\varphi(p\Omega) & p = 2N. \end{cases}$$

Consequently, the noise signal contains contributions only from those components of the spectrum of amplitude fluctuations that correspond in frequency with the odd harmonics of the modulation frequency and from those components of the spectrum of phase fluctuations that correspond in frequency with the even harmonics of the modulation. A more detailed treatment taking into account the terms of order $(T\Gamma_1)^{-1}$ and $(T\Gamma_2)^{-1}$ shows that, in addition to the amplitude fluctuations at frequencies $(2N-1)/\Omega$ and phase fluctuations at frequencies $2N\Omega$, the noise signal contains contributions from their derivatives at these frequencies.

We can now calculate the figure of merit of the double resonance in the system under study,

$$F = G_1[N_\Phi^2 + N_{MW}^2]^{-1/2}T^{-1/2}. \tag{17}$$

Here G_1 and N_{NW}^2 are given by expressions (11) and (16), and the mean square amplitude of the shot noise is proportional to the intensity of the photocurrent, i.e., to the pumping rate:

$$N_\Phi^2 = k^2(W_1 + W_2)T^{-1}. \tag{18}$$

We assume that the shot noise of the photocurrent is small in comparison with the noise due to fluctuations of the rf field. Then the figure of merit can be written in the form

$$F_{MW} = G_1\cdot[N_{MW}^2]^{-1/2}T^{-1/2}. \tag{19}$$

The value of this "real" figure of merit is completely determined by the spectral densities of the fluctuation process at frequencies that are multiples of Ω. In the case of phase fluctuations alone, with the standard values $\eta = \Omega/\Gamma_2 = 0.5$ and $q = 1/(\sqrt{2}\eta) = \sqrt{2}$, expression (19) assumes the simple form

$$F_{MW} \simeq \frac{c_0(\eta)}{\Gamma_2}\left[\frac{1}{\sqrt{S'_\varphi(2\Omega)}} - \frac{c_1(\eta)}{\sqrt{S'_\varphi(4\Omega)}} - \ldots\right], \tag{20}$$

where $c_0(0.5) = 0.66$, $c_1(0.5) = 10^{-3}$ (for $\eta = 1$ and $q = 1/\sqrt{2}$ one has $c_0(1) = 0.91$ and $c_1(1) = 6\cdot 10^{-5}$).

Thus the figure of merit in these cases is determined almost entirely by the spectral density of phase fluctuations at twice the modulation frequency.[1)]

3. In conclusion we shall show that at the present state of the art of producing microwave fields in frequency standards with optical pumping the real values of the figure of merit turn out to be limited by fluctuations of the microwave field. As we have said, the microwave resonance field is usually formed by converting the signals of a highly stable quartz oscillator to the frequency of the working transition of the QFD. The fluctuation characteristics of these oscillators have a standard representation in the form of an Allen diagram; for good oscillators the dependence of the variable of the frequency fluctuatins on the measurement time can be represented for $\tau \lesssim 1$ s in the form $\sigma_\tau^2 = \sigma_0^2\tau^{-1}$, where $\sigma_0 = (1\text{-}5)\cdot 10^{-12}\ s^{1/2}$. such a model of the fluctuations corresponds to white noise with a spectral density σ_0 in a band of 1 s^{-1} in the spectrum of the frequency fluctuations of the oscillator. The rms value of the random modulation index at frequency fluctuations of the oscillator. The rms value of the random modulation index at frequency 2Ω is $q_M = \sigma_0\omega_0/2\Omega$ ($\omega_0 = 2\pi\cdot 6.8\cdot 10^9\ s^{-1}$ for ^{87}Rb). The spectral density of the fluctuations at frequency 2Ω can be determined in terms of q_M: $S'(2\Omega) \approx q_M^2/4$ for $q_M \ll 1$. Then, according to (20),

$$F_{MW} \simeq \frac{4c_0(\eta)}{\sigma_0\omega_0}\eta\,[s^{1/2}] = \frac{8\pi c_0(\eta)}{\sigma_0\omega_0}\eta\,[\mathrm{Hz}^{-1}\cdot s^{-1/2}]. \tag{21}$$

Thus, when the oscillator has a white spectrum of frequency noise the "real" figure of merit, limited by fluctuations of the rf field, is a function of the spectral density of frequency fluctuations σ_0 and the ratio $\eta = \Omega/\Gamma_2$. At a fixed value of η (ordinarily $\eta = 0.5\text{-}1.0$) the "real" figure of merit depends on neither the signal amplitude nor the width of the resonance but is completely determined by the stability of the reference oscillator.[2)]

For $\eta = 0.5$ and $\sigma_0 = 3\cdot 10^{-12}\ s^{1/2}$ the figure of merit calculated according to (21) turns out to be limited by the value $F_{MW} \approx 60\ \mathrm{Hz}^{-1}s^{-1/2}$. On the other hand, it has been shown[5,14,15] that the limiting (limited by photocurrent noise) figure of merit for a rubidium frequency discriminator with optical pumping reaches $10^3\ \mathrm{Hz}^{-1}s^{-1/2}$. According to the above estimate, this value cannot be realized with the existign methods of microwave field generation; according to (20), in order to achieve it the spectral density of phase fluctuations at frequency 2Ω would have to obey $S'(2\Omega) \lesssim 10^{-11}$, which corresponds to a suppression of the phase fluctuations in the spectrum of the microwave oscillator at the level $-(110\text{-}120)$ dB with respect to the carrier.

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[1)]This result indicates that our problem is related to the familiar effect wherein error appears in the determination of the frequency in a QFD in the presence of even harmonic signals modulating the rf field. Result (20) should be understood as the rms shift of the zero of the error signal for a uniform random distribution of the phase of the even harmonics of the modulation.

The quantity $dS_z(x) = E\{ds_z ds_z^*\}$ describes the spectral density of fluctuations of the field in the neighborhood of the regular frequency $\omega + x$, and in general by summation we mean the Lebesgue sum.[8] This definition of the spectral densities avoids the formal difficulties that ordinarily arise in going from stationary random processes to their spectra.[9]

The rf field used to excite double resonance in QFDs is ordinarily formed by multiplication and synthesis of the frequency corresponding to the working transition from the signals of a highly stable quartz oscillator. There are many models describing the fluctuation spectra of these oscillators and the noise introduced by the synthesis procedure.[10] In the case investigated here, condition (2) restricts the phase–frequency fluctuation processes to those with a limited variation of the phase; this permits a substantial simplification of the analysis without loss of generality. Ultimately we will be interested in the parameters of the fluctuation spectrum in a narrow neighborhood of several fixed frequencies lying in a band with a width of the order of that of the working resonance. As we shall show below, these parameters completely determine the noise properties of the discriminator signal, regardless of the manner in which the rf field spectrum is formed.

2. We consider the response of an atomic system whose ground state is split into two sublevels $|1\rangle$ and $|2\rangle$ to the combined action of an optical pump and an rf field of the form

$$U = U_0[1 + z(t)]\, e^{i\omega t + i\Phi_M} + \text{c. c.}, \tag{6}$$

where $\Phi_M = q \sin\Omega t$.

The evolution of the density matrix elements ρ_{ik} of such a system is described by equations of the form

$$\begin{aligned} i\hbar\dot\rho_{11} &= -\rho_{12}H_{21} + \rho_{21}H_{12} - i\hbar\rho_{11}(W_1 + \gamma_1) + i\hbar R/2,\\ i\hbar\dot\rho_{22} &= -\rho_{21}H_{12} + \rho_{12}H_{21} - i\hbar\rho_{22}(W_2 + \gamma_1) + i\hbar R/2,\\ i\hbar\dot\rho_{12} &= -\rho_{12}(H_{22} - H_{11}) + H_{12}(\rho_{22} - \rho_{11}) - i\hbar\rho_{12}[(W_1 + W_2)/2 + \gamma_2],\\ R &= W_1\rho_{11} + W_2\rho_{22} + \gamma_1,\\ \rho_{11} &+ \rho_{22} = 1, \end{aligned} \tag{7}$$

where $H_{ik} = \langle i|\mu U/h|k\rangle$ is the matrix element of the interaction with the rf field; W_1 and W_2 are the rates of optical excitation of the levels $|1\rangle$ and $|2\rangle$, respectively; γ_1 is the rate of dipole relaxation of the levels $|1\rangle$ and $|2\rangle$; γ_2 is the rate of transverse relaxation; R is the number of transitions undergone by an atom per unit time.

We assume that there is no coherence of the ground and higher-lying states and that the excited atom has equal probabilities of decaying to levels $|1\rangle$ and $|2\rangle$ (the case of complete collisional mixing of the excited states). The signal observed in the case of optical detection of the double resonance is proportional to the change in the number of photons absorbed by an atom per unit time under the influence of the rf resonance field, $\tilde{A}(t, \delta) = R - R^{(0)}$. In the framework of the assumptions made above, the solution of system (7) to second order of perturbation is

$$\begin{aligned} \tilde{A}(t, \delta) = B \sum_m \sum_n J_m(q)\, J_{m-n}(q) \Big[&\frac{e^{in\Omega t}}{(in\Omega + \Gamma_1)(i\delta_m + \Gamma_2)}\\ &+ \int \frac{ds^*(x)\, e^{i(n\Omega - x)t}}{(i(n\Omega - x) + \Gamma_1)(i\delta_m + \Gamma_2)}\\ &+ \int \frac{ds(x)\, e^{i(n\Omega + x)t}}{(i(n\Omega + x) + \Gamma_1)(i(\delta_m + x) + \Gamma_2)} + \text{c. c.}\Big], \end{aligned} \tag{8}$$

where

$$\Gamma_i = (W_1 + W_2)/2 + \gamma_i \quad (i = 1,\ 2), \quad \delta_m = \delta + m\Omega,$$

$$\delta = \omega - \omega_0, \quad \omega_0 = (H_{22} - H_{11})/h,$$

$$B = \frac{(W_1 - W_2)^2}{2\Gamma_1} \left| \frac{\mu_{12}U_0}{h} \right|^2.$$

The first term in (8) corresponds to the regular part of the signal and agrees with the results of Refs. 11-13. Ordinarily the component at the modulation frequency Ω is selected out from the signal (8) in the frequency measurement circuitry. By synchronous (with the modulation) detection this component is converted to signals corresponding to the detuning in the zero-frequency region. The spectrum of the converted signal is limited to an effective bandwidth $\pm T^{-1}$ by means of an integrating filter, with $T^{-1} \ll \Gamma_1, \Gamma_2, \Omega$.

The operation of synchronous detection requires multiplication of the signal (8) by $\sin(\Omega t + \phi_0)$ (ϕ_0 is the phase of the synchronous detection) and selection of the component in the neighborhood of zero frequency. After the corresponding transformations one can write

$$A = A_r(\delta) + A_N(\delta, ds_z, x, T). \tag{9}$$

At small detunings, with the phase of synchronous detection chosen as $\phi_0 = -\tan^{-1}(\Omega/\Gamma_1)$, the regular error signal can be written as

$$A_r(\delta) \simeq -G_1 \cdot \delta, \tag{10}$$

where

$$G_1 = \frac{dA_r}{d\delta}\bigg|_{\delta=0} = \frac{B}{\sqrt{\Omega^2 + \Gamma_1^2}} \sum_m J_m(J_{m-1} - J_{m+1}) \frac{\Gamma_2^2 - m^2\Omega^2}{(\Gamma_2^2 + m^2\Omega^2)^2} \tag{11}$$

is the sensitivity factor of the discriminator.

The second term in (9) describes the total fluctuation signal, from which in turn one can select out the contributin of the slow fluctuations ($|x| \lesssim T^{-1}$) the dynamic detuning signal $A_{N_0}(\delta, S_z)$. Assuming that the steady-state detuning δ is equal to zero, we obtain

$$E\{A_{N_0}^2(0, S_z)\} = G_1^2 \cdot 4 \int_{-1/T}^{1/T} x^2 dS_\varphi(x). \tag{12}$$

In the calculations we have made use of the symmetry properties of the spectral representations:

$$ds_a(-x) = ds_a^*(x), \tag{13}$$

$$ds_\varphi(-x) = ds_\varphi^*(x), \tag{14}$$

which are a consequence of (4). Thus the informative part of the signal for $\delta = 0$ completely determined by the time derivative of the phase fluctuations in the band $\pm T^{-1}$, the variance of which has a spectral representation of the form $\int_{-1/T}^{1/T} x^2 dS_\varphi(x)$. This derivative plays the role of a dynamic detuning of the rf field, hile the quantity $A_{N_0}(0, S_z)$ is the dynamic analog of the error signal (10). The presence of a finite detuning δ leads to additional terms of the following form in the expression for the signal:

$$E\{A_{N_0}^2(\delta, S_z)\} = G_1^2 \cdot \delta^2 \cdot 4 \int_{-1/T}^{1/T} dS_a(x). \tag{15}$$

2) This is valid as long as $N_\Phi^2/N_{MW}^2 \ll 1$. The results of a study of the dependence of the figure of merit on the parameters of the pump and the characteristics of the noise in the signal channel over the entire range of variation of the ratio N_Φ^2/N_{MW}^2 are given in Ref. 14.

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